University of Manchester Institute of Science and Technology

UA351: Pure Topics II UA351

For candidates taking:

DEGREE OF MMath
DEGREE OF BSc
FINAL EXAMINATION in HONOURS SCHOOL OF
PURE MATHEMATICS AND COMPUTATION
MATHEMATICS AND LANGUAGE STUDIES
MATHEMATICS
MATHEMATICS STATISTICS AND OPERATIONAL RESEARCH
MATHEAMATICS AND MANAGEMENT SCIENCES

Date: ???????? Time: ????

Answer Three Questions

1.

(i) Given that the set of all 3×3 orthogonal real matrices with determinant $+1$ forms a group $SO(3)$ under matrix multiplication, prove that $SO(3)$ has a subgroup R_Z consisting of rotations about the z-axis. Write down without proof the definitions of two other subgroups, R_X, R_Y, of $SO(3)$, corresponding to rotations about the x-axis and y-axis respectively.

(ii) Prove that the map

$$g : [0, 2\pi] \times [-\pi/2, \pi/2] \to \mathbb{E}^3 : (u, v) \mapsto (\cos v \cos u, \cos v \sin u, \sin v)$$

provides a parametrization of the standard unit sphere $\mathbb{S}^2$.

(iii) Prove that R_Z defines an action on the standard unit sphere $\mathbb{S}^2$ by the map

$$\rho : R_Z \times \mathbb{S}^2 \to \mathbb{S}^2 : ([A_{ij}], [p_j]) \mapsto [\sum_{k=1}^{k=3} A_{ik} p_k]$$

and find its orbits. Prove also that this action is effective but neither free nor transitive.

(iv) Find a parametric equation for the equator curve on $\mathbb{S}^2$ in the form

$$e : [0, 2\pi] \to \mathbb{S}^2$$

and find the parameter value $t_0 \in [0, 2\pi]$ such that

$$e(t_0) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) = p, \text{ say.}$$

Find a parametric equation for the great circle γ through p and inclined at angle $\pi/6$ to the equator, so that $\gamma(t_1) = p$ for some t_1 in the domain of γ and $\gamma'(t_1)$ makes an angle $\pi/6$ with $e'(t_0)$ at p. What is the torsion of γ?

2.

Let $\alpha : (a, b) \to \mathbb{E}^3$ be a regular curve with positive curvature κ, torsion τ and a Frenet-Serret frame field (T, N, B).

Denote by $s : (a, b) \to \mathbb{R}$ the arc length function and suppose that $\hat{\alpha} : (c, d) \to \mathbb{E}^3$ is a unit speed reparametrization of α.

Denote by $\hat{\kappa}$ and $\hat{\tau}$ the curvature and torsion respectively of $\hat{\alpha}$, and by $(\hat{T}, \hat{N}, \hat{B})$ the Frenet-Serret frame field of $\hat{\alpha}$.

(i) Write down without proof the Frenet-Serret equations for $\hat{\alpha}$ and use the definitions

$$\begin{aligned} \kappa(t) &= \hat{\kappa}(s(t)) \\ \tau(t) &= \hat{\tau}(s(t)) \\ T(t) &= \hat{T}(s(t)) \\ N(t) &= \hat{N}(s(t)) \\ B(t) &= \hat{B}(s(t)) \end{aligned}$$

to obtain the Frenet-Serret equations for α.

(ii) Prove that α has acceleration

$$\alpha'' = s''T + (s')^2 \kappa N$$

with curvature

$$\kappa = \frac{||\alpha' \times \alpha''||}{||\alpha'||^3}$$

and torsion

$$\tau = \frac{\alpha' \times \alpha'' \cdot \alpha'''}{||\alpha' \times \alpha''||^2}.$$

(iii) Find the curvature and torsion of the twisted cubic curve

$$\alpha : (0, 1) \to \mathbb{E}^3 : t \mapsto (t, t^2, t^3).$$

3.

Let $\Phi : M_1 \to M_2$ be a local isometry between two regular surfaces M_1 and M_2 in $\mathbb{E}^3$ with Gaussian curvatures K_1, K_2, respectively. A famous theorem of Gauss states that then

$$K_1 = K_2 \circ \Phi.$$

Prove by counterexample that the converse is false by considering the funnel surface M_1 with patch map

$$(u, v) \mapsto (v \cos u, v \sin u, \log v)$$

the helicoid M_2 with patch map

$$(u, v) \mapsto (v \cos u, v \sin u, u)$$

and the diffeomorphism

$$\Phi : M_1 \to M_2 : (v \cos u, v \sin u, \log v) \mapsto (v \cos u, v \sin u, u).$$

Do this by showing that this M_1 and M_2 have the same curvature at corresponding points under Φ, but Φ is not an isometry. You may use without proof the Weingarten result that the determinant of the shape operator S is given by

$$\det S = \frac{eg - f^2}{EG - F^2}$$

when the arc length formula is

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

and, for a unit normal field $\hat{n}$ and a patch map x, the second fundamental form has components given by

$$e = \hat{n} \cdot x_{uu}, \quad f = \hat{n} \cdot x_{uv}, \quad g = \hat{n} \cdot x_{vv}.$$

4.

(i) Draw a projection of an oriented trefoil knot T, and number the overcrossings. From your drawing, write down without proof a matrix from which the Alexander polynomial $\mathcal{A}(T)$ may be found as the monic factor of a determinant. Find $\mathcal{A}(T)$.

(ii) Take an identical pair of oriented trefoil knots and by joining them together form an oriented sum knot, R; take two more oriented trefoil knots, one the mirror image of the other, and join them to form a different summand oriented knot, G; do this in such a way that R and G each have a projection with six overcrossings. [In fact, one of your summands should be the square knot and the other should be the granny knot.] Find suitable such projections, number the overcrossings and then compute for R, G without proof the Alexander polynomials $\mathcal{A}(R)$ and $\mathcal{A}(G)$.

(iii) Write down an equation relating the three polynomials, $\mathcal{A}(T)$, $\mathcal{A}(R)$ and $\mathcal{A}(G)$.